The structure of the clones of self-dual functions in three-valued logic

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***Abstract*—The lattice of all clones of self-dual functions in three-valued logic is described in the paper. Even though this lattice contains a continuum of clones, a simple description was found. Using this description different properties of the lattice and of the clones were derived. Pairwise embeddability of the clones into each other was described, and all finitely generated clones were found. Also, for each clone the relation degree, the cardinality of the set of all clones containing this clone, and the cardinality of the set of all clones that are contained in this clone were determined.**

I. Introduction

In [1], [2] Post described all clones in two-valued logic. It turned out that all clones are finitely generated and the lattice of all clones is countable. In 1959 it was proved that there exists a continuum of clones in three-valued logic [3]. Jablonskij [4] described all maximal (also known as precomplete) classes in three-valued logic. It turned out that all maximal classes except the class of all linear functions contain a continuum of subclasses. In particular, Marchenkov [5], [6] showed that there exists a continuum of subclasses of the maximal class of self-dual functions (which consists of all functions that preserve the relation $\begin{pmatrix}0&1&2\\1&2&0\end{pmatrix}$). In spite of continuum cardinality we found a simple description of all clones of self-dual functions, which is presented in this paper.

We define one finite class of clones, one countable class of clones, and one continual class of clones. Using this description we show different properties of the lattice and of the clones. Firstly, we describe pairwise embeddability of the clones into each other. For the finite and countable classes of clones pairwise embeddability is shown by a graph in figure 1. For the continual class of clones we formulate theorems that describe pairwise embeddability. Then, the relation degree of each clone is found and all finitely generated clones are described. Finally, for each clone we find the cardinality of the set of all clones containing this clone and the cardinality of the set of all clones that are contained in this clone.

II. Main definitions

Most of the notation is taken from book [7]. Let $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$, $E_3 = \{0,1,2\}$. Denote $P_3^n = \{f^n | f^n : E_3^n \to E_3\}$ for $n \geqslant 1$, $P_3 = \bigcup\limits_{n\geqslant 1} P_3^n$. Suppose $F \subseteq P_3$, then by $[F]$ we denote the closure of F under superposition. A set $F \subseteq P_3$ is called a clone if F is closed and contains all projections. By J_3 we denote the set of all projections. The clones form an algebraic lattice whose least element is J_3 and whose greatest element is P_3.

A mapping $E_3^h \to \{0,1\}$ is called an h-ary relation. Let

$$R_3^h = \{\rho | \rho : E_3^h \to \{0,1\}\}, R_3 = \bigcup_{h\geqslant 1} R_3^h.$$

In this paper the relation is often written as a matrix. We write

$$\rho = \begin{pmatrix} b_{1,1} & b_{2,1} & \ldots & b_{n,1}\\ b_{1,2} & b_{2,2} & \ldots & b_{n,2}\\ \ldots & \ldots & \ldots & \ldots\\ b_{1,h} & b_{2,h} & \ldots & b_{n,h}\end{pmatrix}$$

if $\rho \in R_3^h$, $\rho(b_{i,1}, b_{i,2}, \ldots, b_{i,h}) = 1$ for every i and the relation ρ is equal to 0 on the other tuples. We say that

$$\begin{pmatrix} b_1\\ b_2\\ \ldots\\ b_h\end{pmatrix} \in \rho \text{ if } \rho(b_1, b_2, \ldots, b_h) = 1.$$

A function $f \in P_3^m$ preserves a relation ρ if

$$f\begin{pmatrix} a_{1,1} & \ldots & a_{m,1}\\ a_{1,2} & \ldots & a_{m,2}\\ \ldots & \ldots & \ldots\\ a_{1,h} & \ldots & a_{m,h}\end{pmatrix} := \begin{pmatrix} f(a_{1,1},\ldots,a_{m,1})\\ f(a_{1,2},\ldots,a_{m,2})\\ \ldots\\ f(a_{1,h},\ldots,a_{m,h})\end{pmatrix} \in \rho$$

for every

$$\begin{pmatrix} a_{1,1}\\ a_{1,2}\\ \ldots\\ a_{1,h}\end{pmatrix}, \begin{pmatrix} a_{2,1}\\ a_{2,2}\\ \ldots\\ a_{2,h}\end{pmatrix}, \ldots, \begin{pmatrix} a_{m,1}\\ a_{m,2}\\ \ldots\\ a_{m,h}\end{pmatrix} \in \rho.$$

By $Pol(\rho)$ we denote the set of all functions $f \in P_3$ that preserve the relation ρ. For $S \subseteq R_3$ we put

$$Pol(S) = \bigcap_{\rho\in S} Pol(\rho).$$

III. The structure of the clones

Now we define several relations, which we are going to use to define the three classes of clones.

$$\rho_{+1} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \rho_T = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix},$$

$$\rho_N = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}, \rho_W = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{pmatrix},$$

$$\rho_Q = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \end{pmatrix},$$

$$\rho_L(x_1, x_2, x_3) = 1 \iff x_1 + x_2 = 2x_3 (mod\ 3),$$

$$\rho_{L2}(x_1, x_2, x_3, x_4) = 1 \iff (\forall i\ x_i \in \{0,1\}) \wedge \wedge (x_1 + x_2 = x_3 + x_4 (mod\ 2)),$$

$$\rho_{\vee,n}(x_1, \dots, x_n) = 1 \iff (\forall i\ x_i \in \{0,1\}) \wedge \wedge ((x_1 = 1) \vee (x_2 = 1) \vee \dots \vee (x_n = 1)),$$

$$\rho_{\wedge,n}(x_1, \dots, x_n) = 1 \iff (\forall i\ x_i \in \{0,1\}) \wedge \wedge ((x_1 = 0) \vee (x_2 = 0) \vee \dots \vee (x_n = 0)),$$

$$\rho_{=,01}(x_1, x_2, x_3) = 1 \iff (x_1 = 1) \vee ((x_1 = 0) \wedge \wedge (x_2, x_3 \in \{0,1\}) \wedge (x_2 = x_3)),$$

$$\rho_{=,10}(x_1, x_2, x_3) = 1 \iff (x_1 = 0) \vee ((x_1 = 1) \wedge \wedge (x_2, x_3 \in \{0,1\}) \wedge (x_2 = x_3)),$$

$$\rho_{=,012}(x_1, x_2, x_3) = 1 \iff x_1 = 1 \vee (x_1 = 0 \wedge x_2 = x_3),$$

$$\rho_{=,102}(x_1, x_2, x_3) = 1 \iff x_1 = 0 \vee (x_1 = 1 \wedge x_2 = x_3).$$

Class Θ of clones.

$$\mathbf{S} = Pol(\{\rho_{+1}\}), \mathbf{S_0} = Pol(\{\rho_{+1}, (0)\}),$$

$$\mathbf{SL} = Pol(\{\rho_{+1}, \rho_L\}), \mathbf{1S} = [\{(x+1)(mod 3)\}],$$

$$\mathbf{SL_0} = Pol(\{\rho_{+1}, \rho_L, (0)\}), \mathbf{T} = Pol(\{\rho_{+1}, \rho_T\}),$$

$$\mathbf{C} = Pol\left(\left\{\rho_{+1}, \begin{pmatrix} 0 & 1 \end{pmatrix}\right\}\right), \mathbf{D} = Pol\left(\left\{\rho_{+1}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}\right),$$

$$\mathbf{M} = Pol\left(\left\{\rho_{+1}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\right\}\right), \mathbf{DM} = \mathbf{D} \cap \mathbf{M},$$

$$\mathbf{DN} = Pol(\{\rho_{+1}, \rho_N, \rho_N^*\}),$$

$$\mathbf{TD} = \mathbf{T} \cap \mathbf{D}, \mathbf{TM} = \mathbf{T} \cap \mathbf{M}, \mathbf{TN} = \mathbf{DN} \cap \mathbf{T},$$

$$\mathbf{L} = Pol(\{\rho_{+1}, \rho_{L2}\}), \mathbf{TL} = \mathbf{L} \cap \mathbf{T},$$

$$\mathbf{C_2} = \mathbf{L} \cap \mathbf{M}, \mathbf{TC_2} = \mathbf{C_2} \cap \mathbf{T}, \mathbf{O} = [\{x\}].$$

Class Φ of clones.
For $n \geqslant 2$

$$\mathbf{a_n} = Pol(\{\rho_{+1}, \rho_{\vee,n}\}), \mathbf{A_n} = Pol(\{\rho_{+1}, \rho_{\wedge,n}\}),$$

$$\mathbf{a_n M} = \mathbf{a_n} \cap \mathbf{M}, \mathbf{A_n M} = \mathbf{A_n} \cap \mathbf{M},$$

$$\mathbf{a_n N} = Pol(\{\rho_{+1}, \rho_{\vee,n}, \rho_N\}),$$

$$\mathbf{A_n N} = Pol(\{\rho_{+1}, \rho_{\wedge,n}, \rho_N^*\}),$$

$$\mathbf{a_\infty} = \bigcap_n \mathbf{a_n}, \mathbf{A_\infty} = \bigcap_n \mathbf{A_n},$$

$$\mathbf{a_\infty M} = \bigcap_n \mathbf{a_n M}, \mathbf{A_\infty M} = \bigcap_n \mathbf{A_n M},$$

$$\mathbf{a_\infty N} = \bigcap_n \mathbf{a_n N}, \mathbf{A_\infty N} = \bigcap_n \mathbf{A_n N},$$

$$\mathbf{aP} = Pol(\{\rho_{+1}, \rho_Q\}),$$

$$\mathbf{AP} = Pol\left(\{\rho_{+1}, \rho_Q^*\}\right),$$

$$\mathbf{aPN} = Pol(\{\rho_{+1}, \rho_Q, \rho_N\}),$$

$$\mathbf{APN} = Pol\left(\{\rho_{+1}, \rho_Q^*, \rho_N^*\}\right),$$

$$\mathbf{aP_1} = Pol(\{\rho_{+1}, \rho_Q, \rho_W\}),$$

$$\mathbf{AP_1} = Pol\left(\{\rho_{+1}, \rho_Q^*, \rho_W^*\}\right).$$

For $n \geqslant 2$

$$\mathbf{aP_n} = \mathbf{aP_1} \cap Pol(\pi_{\{1,2,\dots,n\}}),$$

$$\mathbf{AP_n} = \mathbf{AP_1} \cap Pol(\pi^*_{\{1,2,\dots,n\}}),$$

$$\mathbf{aP_\infty} = \bigcap_n \mathbf{aP_n}, \mathbf{AP_\infty} = \bigcap_n \mathbf{AP_n},$$

$$\mathbf{aQ} = Pol(\{\rho_{+1}, \rho_{=,01}\}), \mathbf{AQ} = Pol(\{\rho_{+1}, \rho_{=,10}\}),$$

$$\mathbf{aW} = Pol(\{\rho_{+1}, \rho_{=,012}\}), \mathbf{AW} = Pol(\{\rho_{+1}, \rho_{=,102}\}).$$

We will need the following notation to define the last and the most complicated class of clones. Let $m \in \mathbb{N}$, $n \in \mathbb{N}_0$. By D_n^m we denote the set of all tuples $(A_1, \dots, A_m)$ such that $A_1, \dots, A_m \subseteq \{1, 2, \dots, n\}$, $A_1 \cup \dots \cup A_m = \{1, 2, \dots, n\}$. (In case of $n = 0$ we have $A_1 = A_2 = \dots = A_m = \varnothing$).

Let $D = \bigcup_{3 \leqslant m+n \leqslant l} D_n^m$.

Let us define several binary relations on the set D. Let

$$(A_1, \dots, A_m) \in D_n^m, (A'_1, \dots, A'_{m'}) \in D_{n'}^{m'}.$$

Relation $\simeq$. Let

$$(A'_1, \dots, A'_{m'}) \simeq (A_1, \dots, A_m)$$

iff $m' = m$, $n' = n$, and there exists a permutation $\sigma : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ such that $A'_i = \sigma(A_i)$ for every $i \in \{1, 2, \dots, m\}$.

Relation $\lesssim^1$. Let

$$(A'_1, \dots, A'_{m'}) \lesssim^1 (A_1, \dots, A_m)$$

iff $m' \geqslant m$, $n' \leqslant n$, $m'+n' = m+n$, $A'_i = A_i \cap \{1, 2, \dots, n'\}$ for $i \in \{1, 2, \dots, m\}$, $A'_i = \varnothing$ for $i \in \{m+1, m+2, \dots, m'\}$.

Relation $\lesssim^2$. Let

$$(A'_1, \dots, A'_{m'}) \lesssim^2 (A_1, \dots, A_m)$$

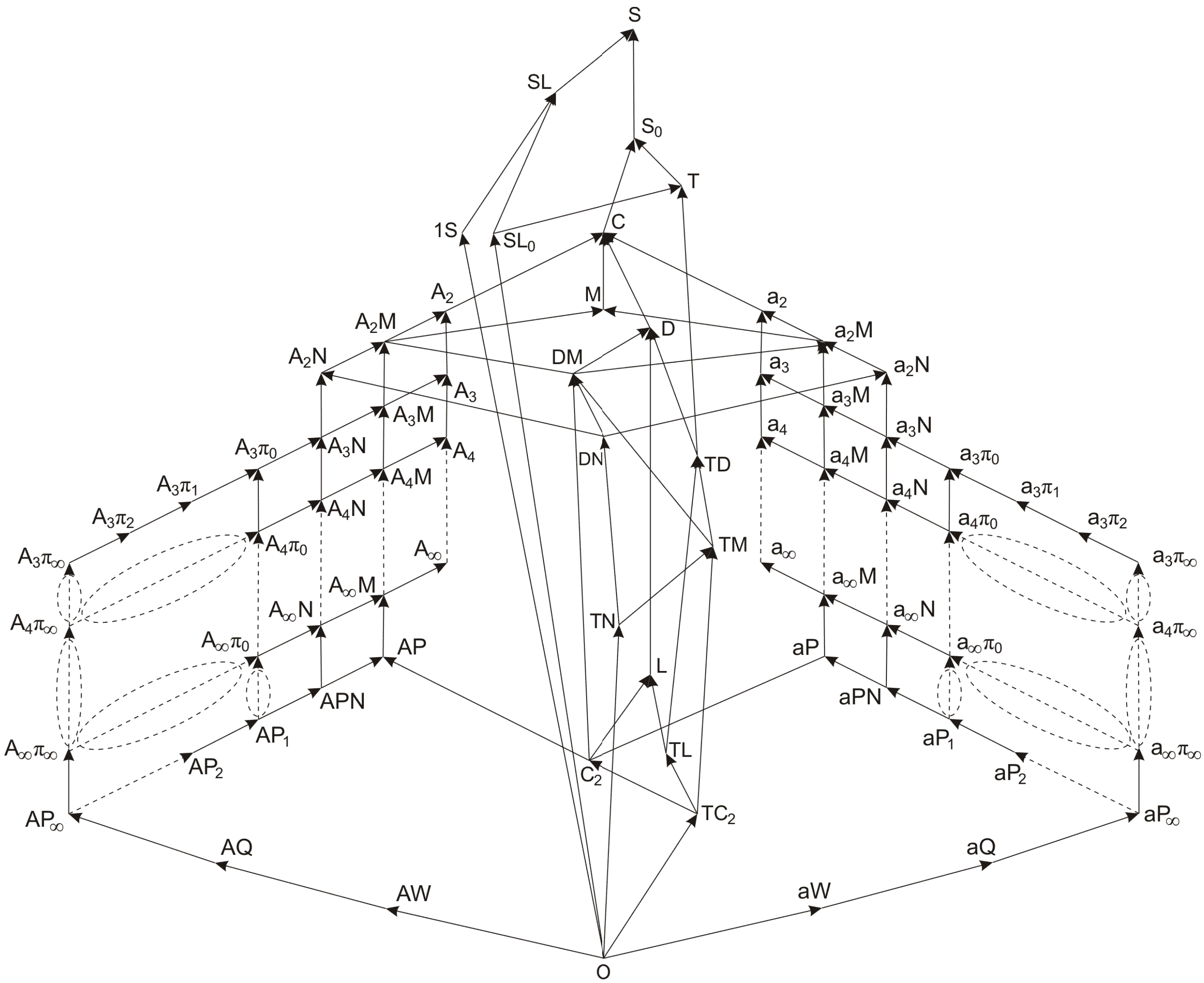

Figure 1: The structure of the clones

iff $m' \leqslant m$, $n' = n$, the set $\{1, 2, \ldots, m\}$ can be divided into non-overlapping sets $K_1, \ldots, K_{m'}$ such that

$$A'_i = \bigcup_{j \in K_i} A_j.$$

Relation $\lesssim^3$. Let

$$(A'_1, \ldots, A'_{m'}) \lesssim^3 (A_1, \ldots, A_m)$$

iff $m' = m$, $n' = n$, $A_i \subseteq A'_i$ for every $i \in \{1, 2, \ldots, m\}$.

Relation $\lesssim$. Suppose $\Omega, \Omega' \in D$, then put $\Omega' \lesssim \Omega$ iff

$$\exists \Omega_1, \Omega_2, \Omega_3\ (\Omega' \lesssim^3 \Omega_3 \wedge \Omega_3 \lesssim^2 \Omega_2 \wedge \Omega_2 \lesssim^1 \Omega_1 \wedge \Omega_1 \simeq \Omega).$$

Lemma 1. *Binary relation $\lesssim$ is transitive and reflexive.*

It can be shown that

$$(\varnothing, \varnothing, \varnothing) \lesssim (A_1, \ldots, A_m)$$

for every $(A_1, \ldots, A_m) \in D$.

To each $(A_1, \ldots, A_m) \in D_n^m$ we assign the relation $\pi_{A_1,\ldots,A_m} \in R^{m+n}$ such that

$$\pi_{A_1,\ldots,A_m}(x_1, \ldots, x_m, y_1, \ldots, y_n) = 1$$

iff the following conditions hold:

1) $\forall i (x_i = 1 \vee (x_i = 0 \wedge (\forall j \in A_i :\ y_j \in \{0, 1\})))$;

2) at least one of the values $x_1, \ldots, x_m, y_1, \ldots, y_n$ is not equal to 0.

By Π_n^m we denote the set of all relations $\pi_{A_1,\ldots,A_m} \in R_3^{m+n}$ such that

$$(A_1, \ldots, A_m) \in D_n^m.$$

Put $\Pi^l = \bigcup\limits_{3 \leqslant m+n \leqslant l} \Pi_n^m, \Pi_l = \bigcup\limits_{n \leqslant l, m+n \geqslant 3} \Pi_n^m, \Pi = \bigcup\limits_{l} \Pi^l$. It can be easily shown that we have one-to-one correspondence between elements of D_n^m and elements of Π_n^m.

The binary relation $\lesssim$ on the set Π is defined as follows

$$\pi_{A'_1,\ldots,A'_{m'}} \lesssim \pi_{A_1,\ldots,A_m} \iff (A'_1,\ldots,A'_{m'}) \lesssim (A_1,\ldots,A_m).$$

Let $\sigma : E_3 \longrightarrow E_3$, $\sigma(0) = 1$, $\sigma(1) = 0$, $\sigma(2) = 2$. By ρ^* we denote the relation that is dual to ρ :

$$\rho^*(x_1,\ldots,x_n) := \rho(\sigma(x_1),\sigma(x_2),\ldots,\sigma(x_n)).$$

Suppose $S \subseteq R$, then put

$$S^* := \{\rho | \rho^* \in S\}.$$

We say that a set $F \subseteq \Pi$ is closed under the relation $\lesssim$ if

$$\forall \rho \in F\ \forall \rho' \in \Pi\ (\rho' \lesssim \rho \implies \rho' \in F).$$

By $\widetilde{\Pi}$ we denote the set of all $F \subseteq \Pi$ that are not empty and closed under the relation $\rho_{\lesssim}$.

For $F \subseteq \Pi$ we put

$$Clone(F) = Pol\left(F \cup \{\rho_{+1}, \rho_W\}\right),$$

$$Clone^*(F) = Pol\left(F^* \cup \{\rho_{+1}, \rho^*_W\}\right).$$

Class Υ of clones. Suppose $F \in \widetilde{\Pi}$, then

$$Clone(F), Clone^*(F) \in \Upsilon.$$

There are no other clones in Υ.

Theorem 1. *Suppose $F_1, F_2 \in \widetilde{\Pi}$, then*

$$Clone(F_1) \subseteq Clone(F_2) \iff F_1 \supseteq F_2.$$

Corollary 1. *Suppose $F_1, F_2 \in \widetilde{\Pi}$ and $F_1 \neq F_2$, then $Clone(F_1) \neq Clone(F_2)$.*

Theorem 2. *Set $\Upsilon \cup \Theta \cup \Phi$ is the set of all clones M such that $M \subseteq Pol\left(\{\rho_{+1}\}\right)$.*

Pairwise embeddability of clones from Θ and Φ into each other is shown schematically by a graph in figure 1. Vertexes of the graph are clones.

Two vertexes M_1 and M_2 of the graph are joined by a solid line and M_1 is located above M_2 iff $M_2 \subset M_1$ and there does not exist a clone M' such that $M_2 \subset M' \subset M_1$. Two vertexes M_1 and M_2 are joined by a dotted line and M_1 is located above M_2 iff $M_2 \subset M_1$ and there exists an infinite sequence of clones $K_1 \supset K_2 \supset K_3 \supset \ldots$ such that the following conditions hold:

1) $K_1 \subset M_1$;
2) $\bigcap\limits_i K_i = M_2$;
3) if $M_2 \subset M' \subset M_1$, then $M' = K_i$ for some i.

In some other cases a dotted line is located inside a dotted ellipse. So if a dotted line between M_1 and M_2 is located inside a doted ellipse and M_1 is located above M_2 then this means that $M_2 \subset M_1$ but this situation does not satisfy conditions of solid line and dotted line.

Also some clones from the class Υ are shown on the figure 1. For $n \geqslant 3$ put

$$\mathbf{a_n}\pi_{\mathbf{0}} = Clone(\Pi^n \cap \Pi_0), \mathbf{A_n}\pi_{\mathbf{0}} = Clone^*(\Pi^n \cap \Pi_0),$$

$$\mathbf{a_n}\pi_\infty = Clone(\Pi^n), \mathbf{A_n}\pi_\infty = Clone^*(\Pi^n),$$

$$\mathbf{a_\infty}\pi_{\mathbf{0}} = Clone(\Pi_0), \mathbf{A_\infty}\pi_{\mathbf{0}} = Clone^*(\Pi_0),$$

$$\mathbf{a_\infty}\pi_\infty = Clone(\Pi), \mathbf{A_\infty}\pi_\infty = Clone^*(\Pi),$$

$$\mathbf{a_3}\pi_{\mathbf{1}} = Clone(\pi_{\{1\},\{1\}}), \mathbf{A_3}\pi_{\mathbf{1}} = Clone^*(\pi_{\{1\},\{1\}}),$$

$$\mathbf{a_3}\pi_{\mathbf{2}} = Clone(\pi_{\{1\},\varnothing}), \mathbf{A_3}\pi_{\mathbf{2}} = Clone^*(\pi_{\{1\},\varnothing}).$$

The next three theorems describe the embeddability of clones from Υ into clones from Φ and clones from Φ into clones from Υ.

Theorem 3. *Suppose $l \geqslant 3$, $F \in \widetilde{\Pi}$, then $Clone(F) \subset \mathbf{a_l N}$ iff $l = 3$ or $F \not\subseteq \Pi^{l-1}$.*

Theorem 4. *Suppose $F \in \widetilde{\Pi}$, then $\mathbf{aP_l} \subset Clone(F)$ iff $F \subseteq \Pi_l$.*

Theorem 5. *Suppose $F \in \widetilde{\Pi}$, then $\mathbf{aP}_\infty \subset Clone(F)$.*

IV. Some properties of clones from Θ, Φ, and Υ

A clone $M \subseteq P_3$ is called finitely generated if there exists a finite set $M_0 \subseteq M$ such that $M = [M_0]$. The relation degree $d(A)$ of a clone $A \subseteq P_3$ is the smallest $h \in \mathbb{N}$ such that $A = Pol(S)$ for some $S \subseteq R_3^h$, i.e.,

$$d(A) = min\{h | \exists Q \subseteq R_3^h : Pol(Q) = A\}$$

Put $d(A) = \infty$ if $\forall Q \subseteq R_k (A = Pol(Q) \Rightarrow |Q| = \infty)$. Further, let $\mathbb{L}_3$ be the set of all clones in P_3. For $F \in \mathbb{L}_3$ we put

$$\mathbb{L}_3^\uparrow(F) := \{F' \in \mathbb{L}_3 | F \subseteq F'),$$

$$\mathbb{L}_3^\downarrow(F) := \{F' \in \mathbb{L}_3 | F' \subseteq F).$$

Theorem 6. *All clones in the classes Θ and Φ are finitely generated.*

Suppose $\rho_1, \rho_2 \in \Pi$, then put

$$\rho_1 < \rho_2 \iff (\rho_1 \lesssim \rho_2) \wedge (\neg(\rho_2 \lesssim \rho_1)).$$

Suppose $F \in \widetilde{\Pi}$, then put

$$Bound(F) := \{\rho \in \Pi | \rho \notin F, \forall \sigma \in \Pi (\sigma < \rho \Longrightarrow \sigma \in F)\}.$$

Theorem 7. *Suppose $F \in \widetilde{\Pi}$, then $Clone(F)$ is finitely generated iff $Bound(F)$ is finite.*

Corollary 2. *Suppose $F \in \widetilde{\Pi}$, $|F| < \infty$, then $Clone(F)$ is finitely generated.*

Theorem 8. *Suppose $M \in \Theta \cup \Phi$, then*

$$d(M) = \begin{cases} 2, & \textit{if } M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{T}, \mathbf{C}, \mathbf{M}, \mathbf{D}, \mathbf{DM}, \mathbf{DN}, \\ & \mathbf{TD}, \mathbf{TM}, \mathbf{TN}, \mathbf{1S}, \mathbf{O}\}; \\ 3, & \textit{if } M \in \{\mathbf{SL}, \mathbf{SL_0}, \mathbf{L}, \mathbf{TL}, \mathbf{C_2}, \mathbf{TC_2}, \\ & \mathbf{aP}, \mathbf{AP}, \mathbf{aPN}, \mathbf{APN}, \mathbf{aP_1}, \mathbf{AP_1}\}; \\ n, & \textit{if } n \geqslant 2 \textit{ and } M \in \{\mathbf{a_n}, \mathbf{a_n M}, \mathbf{a_n N}, \\ & \mathbf{A_n}, \mathbf{A_n M}, \mathbf{A_n N}\}; \\ n+1, & \textit{if } n \geqslant 2 \textit{ and } M \in \{\mathbf{aP_n}, \mathbf{AP_n}\}; \\ \infty, & \textit{if } M \in \{\mathbf{a_\infty}, \mathbf{a_\infty M}, \mathbf{a_\infty N}, \mathbf{A_\infty}, \\ & \mathbf{A_\infty M}, \mathbf{A_\infty N}, \mathbf{aP_\infty}, \mathbf{AP_\infty}\}; \end{cases}$$

Theorem 9. *Suppose* $F \in \widetilde{\Pi}$, $F \neq \{\pi_{\varnothing,\varnothing,\varnothing}\}$, *then*

$$d(Clone(F)) = \begin{cases} \max\{m+n \mid \exists \rho \in \Pi_n^m \cap F\}, & \text{if } |F| < \infty; \\ \infty, & \text{otherwise.} \end{cases}$$

$d(Clone(\{\pi_{\varnothing,\varnothing,\varnothing}\})) = 2.$

Theorem 10. *Suppose* $M \in \Theta \cup \Phi$, *then*

$$|\mathbb{L}_3^{\downarrow}(M)| \begin{cases} = \aleph_0, & \text{if } M \in \{\mathbf{aP}, \mathbf{aPN}, \mathbf{aP_1}, \mathbf{aP_2}, \mathbf{aP_3}, \dots, \\ & \mathbf{AP}, \mathbf{APN}, \mathbf{AP_1}, \mathbf{AP_2}, \mathbf{AP_3}, \dots\}; \\ = 2^{\aleph_0}, & \text{if } M \in \{\mathbf{S}, \mathbf{S_0}, \mathbf{C}, \mathbf{M}, \mathbf{a}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \\ & \mathbf{A}_\infty, \mathbf{A}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{N}\} \text{ or } M \in \bigcup\limits_{n \geqslant 2} \{\mathbf{a_n}, \mathbf{A_n}, \\ & \mathbf{a_n M}, \mathbf{A_n M}, \mathbf{a_n N}, \mathbf{A_n N}\}; \\ < \infty, & \text{otherwise.} \end{cases}$$

Theorem 11. *Suppose* $F \in \widetilde{\Pi}$, *then*

$$|\mathbb{L}_3^{\downarrow}(Clone(F))| = \begin{cases} 2^{\aleph_0}, & \text{if } F \neq \Pi; \\ 5, & \text{if } F = \Pi. \end{cases}$$

Theorem 12. *Suppose* $M \in \Theta \cup \Phi$, *then*

$$|\mathbb{L}_3^{\uparrow}(M)| \begin{cases} = \aleph_0, & \text{if } M \in \{\mathbf{a}_\infty, \mathbf{A}_\infty, \mathbf{a}_\infty\mathbf{M}, \mathbf{A}_\infty\mathbf{M}, \mathbf{a}_\infty\mathbf{N}, \\ & \mathbf{A}_\infty\mathbf{N}, \mathbf{aP}, \mathbf{aPN}, \mathbf{AP}, \mathbf{APN}, \mathbf{C_2}, \mathbf{TC_2}\}; \\ & \text{or } M \in \bigcup\limits_{n \geqslant 1} \{\mathbf{aP_n}, \mathbf{AP_n}\}; \\ = 2^{\aleph_0}, & \text{if } M \in \{\mathbf{aP}_\infty, \mathbf{AP}_\infty, \mathbf{aQ}, \mathbf{AQ}, \mathbf{aW}, \\ & \mathbf{AW}, \mathbf{O}\}; \\ < \infty, & \text{otherwise.} \end{cases}$$

Theorem 13. *Suppose* $F \in \widetilde{\Pi}$, *then*

$$|\mathbb{L}_3^{\uparrow}(Clone(F))| \begin{cases} = 2^{\aleph_0}, & \text{if } F \text{ contains an infinite set of} \\ & \text{pairwise incomparable(under } \lesssim) \\ & \text{relations;} \\ < \infty, & \text{if } |F| < \infty; \\ = \aleph_0, & \text{otherwise.} \end{cases}$$

Corollary 3. $|\mathbb{L}_3^{\uparrow}(\mathbf{a_n}\pi_\infty)| < \infty$ *for every* $n \geqslant 3$.

It follows from theorem 12 and corollary 3 that $|\mathbb{L}_3^{\uparrow}(\mathbf{aP_m})| = \aleph_0$ for every $m \geqslant 1$, $|\mathbb{L}_3^{\uparrow}(\mathbf{a_n}\pi_\infty)| < \infty$ for every $n \geqslant 3$. Roughly speaking, it means that continuum of clones is located near the vertex $\mathbf{a}_\infty\pi_\infty$ on the figure 1.

REFERENCES

[1] E. Post. *Determination of all closed systems of truth tables.* Bull. Amer. Math. Soc. 26, 427, 1920.

[2] E. Post. *Two-Valued Iterative Systems of Mathematical Logic.* Princeton Univ. Press, Princeton, 1941.

[3] Ju.İ. Janov, A. A. Muchnik. *Existence of k-valued closed classes without a finite basis* (In Russian). Dokl. Akad. Nauk SSSR 127, 1959. N. 1. 44-46.

[4] S. V. Jablonskij. *On functional completeness in the three-valued calculus.* Dokl. Akad. Nauk SSSR 95,1153-1155, 1954.

[5] S. S. Marchenkov, J. Demetrovich, L. Hannak. *On closed classes of self-dual functions in P_3.* (O zamknutih klassah samodvoystvennih funkciy v P_3.) Metody Diskretn. Anal. 34, 38-73 1980 (in Russian).

[6] S. S. Marchenkov. *On closed classes of self-dual functions of many-valued logic II.* (O zamknutih klassah samodvoystvennih funkciy mnogoznachnoy logiki.) Probl. Kibernetiki 40. 261-266, 1983 (in Russian).

[7] D. Lau. *Function algebras on finite sets.* Springer. 2006.